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THE GROUP GENERATED BY CENTRAL SYMMETRIES, WITH APPLICATION TO POLYGONS.

By DR. EDWARD KASNER.

The object of this article is to generalize the following well known theorem of elementary geometry: If the mid-points of the consecutive sides of any quadrilateral are joined the resulting figure is a parallelogram. In the case of a triangle the corresponding construction gives a triangle not having any peculiar property. The question therefore arises as to whether or not in case of polygons of more than four sides there is any theorem analogous to that concerning the quadrilateral. It will be shown that, in this respect, there is an essential distinction between polygons of an even number of sides, and those of an odd number of sides. This depends fundamentally upon the character of the group which is discussed in §1.

§1. THE GROUP.

1. Any translation T of the plane may be written

$$(T) \quad \begin{aligned} x' &= x + h \\ y' &= y + k, \end{aligned}$$

where h and k are the components, in the direction of the coördinate axes, of the vector corresponding to the translation. It is obvious that the totality of translations form a group; for the combination of any two, say T_1 and T_2 , whose vector components are h_1, k_1 and h_2, k_2 respectively, gives

$$\begin{aligned}x' &= x + h_1 + h_2 \\ y' &= y + k_1 + k_2,\end{aligned}$$

which is itself a translation.

2. Consider now the transformations termed *central or point symmetries*. Such a symmetry is defined by

$$(S) \quad \begin{aligned}x' &= -x + 2a \\ y' &= -y + 2b,\end{aligned}$$

where a, b are the coördinates of the center of the symmetry, *i. e.* the fixed point P with respect to which corresponding points x, y and x', y' are symmetric. The symmetries themselves do not form a group, but we now show that

The translation T and the central symmetries S form a group.

In the first place, *the product of two symmetries is a translation*. For, if the center of S_1 is P_1 , with coördinates (a_1, b_1) , and if the center of S_2 is P_2 , with coördinates (a_2, b_2) , the combination $S_1 S_2$ gives

$$\begin{aligned}x' &= x + 2(a_2 - a_1) \\ y' &= y + 2(b_2 - b_1).\end{aligned}$$

The vector of this translation is double the vector $P_1 P_2$. Similarly, the product $S_2 S_1$ is the translation whose vector is twice $P_2 P_1$. In the second place, *the product of a symmetry and a translation is a symmetry*. For, the transformation ST is

$$\begin{aligned}x' &= -x + 2a + h \\ y' &= -y + 2b + k;\end{aligned}$$

this is the symmetry whose center is obtained from the center of S by applying the vector of T . Similarly, the product in the reverse order, *i. e.*, TS is the symmetry whose center is obtained from the center of S by applying the vector opposite to that of T .

It follows then that any combination of transformations T and S is itself either a T or an S , so that the group property is proved. In Lie's terminology the group considered is a mixed two-parameter group consisting of two continuous systems of transformations. The translations constitute a self-conjugate sub-group.

3. Since the product of two symmetries is a translation, and since the translations constitute a group, it follows that *the product of an even number of symmetries is a translation*. The product of the $2k$ symmetries $S_1, S_2, \dots, S_{2k}$ is in fact

$$\begin{aligned}x' &= x + 2(a_{2k} - a_{2k-1} + \dots + a_2 - a_1) \\ y' &= y + 2(b_{2k} - b_{2k-1} + \dots + b_2 - b_1),\end{aligned}$$

where a_i, b_i denote the coördinates of P_i the center of S_i . The formulae may be

interpreted geometrically by observing that the differences $a_2 - a_1$, $b_2 - b_1$, for example, are the components of the vector P_1P_2 ; therefore *the vector of the resulting translation is twice the vector sum*

$$P_1P_2 + P_3P_4 + \dots + P_{2k-1}P_{2k}.$$

4. An odd number of symmetries may be combined by combining the first with the product of all the remaining, which by 3 is a translation. Therefore, *the product of an odd number of symmetries is a symmetry*. If the symmetries are $S_1, S_2, \dots, S_{2k+1}$ their product is

$$x' = -x + 2(a_{2k+1} - a_{2k} + \dots + a_3 - a_2 + a_1)$$

$$y' = -y + 2(b_{2k+1} - b_{2k} + \dots + b_3 - b_2 + b_1).$$

The center of the resulting symmetry is obtained from the center P of the first symmetry by applying the vector sum

$$P_2P_3 + P_4P_5 + \dots + P_{2k}P_{2k+1}.$$

5. The application to be made depends essentially upon the *fixed points* of the transformations, i. e. the points which are transformed into themselves. Excluding points at infinity from consideration, we observe in the first place that in case of a symmetry there is one and but one fixed point, namely, the center of the symmetry. On the other hand, in case of a translation, there are no fixed points, except when the translation reduces to the identical transformation, in which case all the points of the plane are fixed.

§2. MID-POINT POLYGONS.

6. Consider any polygon whose vertices in order may be denoted by $Q_1, Q_2, \dots, Q_n$. If the middle points of the sides are connected in order we derive a new polygon of the same number of sides which for brevity may be termed the *inscribed polygon*; the original polygon in its relation to the derived is then termed the *circumscribed polygon*. For every polygon there is then a definite inscribed polygon. The question now to be considered concerns the converse problem: *Given an arbitrary polygon $P_1, \dots, P_n$, is it possible to construct a circumscribed polygon, i. e., is it possible to find n points $Q_1, Q_2, \dots, Q_n$ such that P_1 shall be mid-way between Q_1 and Q_2 , P_2 mid-way between Q_2 and Q_3 , and so on until finally P_n shall be mid-way between Q_n and P_1 ?*

To answer this question, take tentatively any point Q in the plane; construct with respect to P_1 the symmetric point; then with respect to P_2 construct the point symmetric to the one just obtained; and so in order until finally by symmetry with respect to P_n a point Q' is obtained. The original polygon $P_1, \dots, P_n$ thus defines a definite transformation by which to any point Q corresponds an unique point Q' . This transformation is simply the product of n symmetries

and therefore, by the previous section, is either a translation or a symmetry according as n is even or odd.

7. Consider first the case of a polygon with an odd number of sides $n=2k+1$. The transformation from Q to Q' is then a symmetry. There is, therefore, by §5 a single point which remains invariant under the transformation. Denoting this point by Q_1 , we obtain from it, by successive symmetry with respect to $P_1, P_2, \dots, P_{2k}$, the points $Q_2, Q_3, \dots, Q_{2k+1}$; Q_{2k+1} and Q_1 are then necessarily symmetric with respect to the last vertex P_{2k+1} , so that $Q_1, \dots, Q_{2k+1}$ are in fact the vertices of the unique circumscribed polygon.

Any polygon with an odd number of sides can be obtained as an inscribed polygon; there exists one and only one circumscribed polygon.

The circumscribed polygon may be constructed by applying the result stated at the end of 4. The first vertex Q_1 is obtained from the first vertex P_1 of the original polygon by applying the vector sum $P_2P_3 + P_4P_5 + \dots + P_{2k}P_{2k+1}$; then the remaining vertices $Q_2, \dots, Q_{2k+1}$, are obtained by successive symmetry as described above.

8. If the polygon has an even number of sides $n=2k$, then from 3 the transformation from Q to Q' is a translation which in general does not reduce to the identical transformation. In this case, from 5, there exists no fixed point, and therefore no circumscribed polygon.

In the case of an arbitrary polygon of an even number of sides no circumscribed polygon exists, i. e., not all such polygons can be obtained as inscribed polygons.

9. The construction will, however, be possible in the exceptional case where the resulting translation reduces to identity. If then we term a $2k$ -gon *special* when it is possible to circumscribe a polygon about it, the result may be stated:

Any special $2k$ -gon is characterized by the fact that the product of the symmetries having for centers the vertices of the polygon is identity.

The class of polygons considered may be defined otherwise as follows: From the formulae in 3, we have as the conditions for reducing to identity,

$$\begin{aligned} a_{2k} - a_{2k-1} + \dots + a_2 - a_1 &= 0 \\ b_{2k} - b_{2k-1} + \dots + b_2 - b_1 &= 0; \end{aligned}$$

which together express the vanishing of the vector sum

$$P_1P_2 + P_3P_4 + \dots + P_{2k-1}P_{2k}.$$

The vanishing of this sum necessitates the vanishing of

$$P_2P_3 + P_4P_5 + \dots + P_{2k}P_1,$$

since for any polygon the vector sum of all the sides is zero. Therefore

In any special $2k$ -gon the vector sum of the alternate sides vanish; this condition is also sufficient.

The equation of condition above may also be written

$$\frac{a_1 + a_3 + \dots + a_{2k-1}}{k} = \frac{a_2 + a_4 + \dots + a_{2k}}{k}$$

$$\frac{b_1 + b_3 + \dots + b_{2k-1}}{k} = \frac{b_2 + b_4 + \dots + b_{2k}}{k},$$

which may be interpreted as follows:

In any special $2k$ -gon with vertices $P_1, P_2, \dots, P_{2k}$, the mean point (or center of gravity) of the alternate vertices $P_1, P_3, \dots, P_{2k-1}$ coincides with the mean point of the remaining vertices $P_2, P_4, \dots, P_{2k}$.

Both points obviously coincide with the mean point of all the vertices of the $2k$ -gon.

10. For a special $2k$ -gon the transformation from Q to Q' described in 6 reduces to identity, so that every point of the plane is an invariant point. Therefore in constructing the circumscribed polygon any point may be assumed for the first vertex Q_1 , the other vertices then being determined by successive symmetry with respect to $P_1, P_2, \dots, P_{2k-1}$.

About a special $2k$ -gon a double-infinity of circumscribed $2k$ -gons may be constructed; otherwise stated, if it is possible to circumscribe one polygon about a given $2k$ -gon, it is possible to circumscribe a double infinity.

We shall now prove that among this double infinity of $2k$ -gons there is one which is itself special, so that *about any special $2k$ -gon it is possible to circumscribe one and only one special $2k$ -gon*. Let the first vertex Q_1 of any circumscribed polygon be x, y ; then the next vertex Q_2 , obtained by symmetry with respect to P_1 , is $-x+2a_1, -y+2b_1$; similarly Q_2 is $x-2a_1+2a_2, y-2b_1+2b_2$; finally, Q_{2k} is $-x+2a_1-2a_2+\dots+2a_{2k-1}, -y+2b_1-2b_2+\dots+2b_{2k-1}$. If now the circumscribed polygon is to be special we must have

$$Q_1 Q_2 + Q_3 Q_4 + \dots + Q_{2k-1} Q_{2k} = 0,$$

which is equivalent to

$$kx - (2k-1)a_1 + (2k-2)a_2 - \dots - a_{2k-1} = 0$$

$$ky - (2k-1)b_1 + (2k-2)b_2 - \dots - b_{2k-1} = 0$$

These equations determine x and y , that is, the first vertex Q_1 , uniquely, which proves the theorem announced.

The special quadrilaterals are simply parallelograms. About any parallelogram a double infinity of quadrilaterals may be circumscribed, of which one is itself a parallelogram; about this in turn a parallelogram may be circumscribed and so on indefinitely. So for any special $2k$ -gon, one can not only inscribe special $2k$ -gons indefinitely, but also circumscribe them. Again, just as the quadrilateral inscribed in a general parallelogram is itself an arbitrary parallelogram, so the $2k$ -gon inscribed in a special $2k$ -gon is not further specialized, but is an arbitrary special $2k$ -gon.

11. The second characteristic given in 9 gives the following *construction for special $2k$ -gons*. Take an arbitrary k -gon $D', D'', \dots, D^{(k)}$; on each side construct a parallelogram, thus on $D' D''$ construct $P_1 D_1 D_2 P_2$, on $D'' D'''$ construct $P_3 D'' D''' P_4$, finally construct $P_{2k-1} D^{(k-1)} D^{(k)} P_{2k}$; then $P_1, P_2, \dots, P_{2k}$ will constitute a special $2k$ -gon. To prove this we need merely observe that since the alternate sides $P_1 P_2, P_3 P_4, \dots$, are equal and parallel to $D' D'', D'' D'''$, respectively, the vector sum of the former sides is equal to the vector sum of all the sides of the auxiliary k -gon and therefore vanishes.

It is seen from this that a *special $2k$ -gon is completely determined by $2k-1$ of its vertices*. For if $P_1, P_2, \dots, P_{2k-1}$ are given we can construct the auxiliary k -gon by starting at an arbitrary point D' , drawing the vector $D' D''$ equal to $P_1 P_2$, then $D'' D'''$ equal to $P_3 P_4$, ..., finally, $D^{(k-1)} D^{(k)}$ equal to $P_{2k-3} P_{2k-2}$; P_{2k} is then found by drawing from P_{2k-1} a vector equal to $D^{(k)} D'$. This is the generalization of the fact that a parallelogram is determined by three of its vertices (given of course in order).

After the case $k=2$ of the parallelogram, the first case deserving particular attention is the case $k=3$, *i. e.*, the *special hexagon*. Such a hexagon may be obtained, in accordance with the result above, by constructing parallelograms on the sides of an arbitrary triangle. Another construction is as follows: Take any two parallelograms $ABCO, ODEF$ having a vertex in common; the remaining vertices $ABCDEF$ constitute a special hexagon. The same hexagon may be obtained in this way by means of three distinct pairs of parallelograms. This may be generalized so as to apply to $2k$ -gons.

The third characteristic stated in 9, in the case of a special hexagon $ABCDEF$, shows that the median point of the triangle ACE coincides with that of the triangle BDF ; therefore the six lines obtained by joining each vertex to the mid-point of the opposite diagonal of the hexagon are concurrent, the point of concurrence being the mean point of the hexagon.

§3. EXTENSION TO SPACE.

12. The preceding results admit of immediate extension to space of three dimensions, and to higher spaces. In fact, the translations and central symmetries still constitute a group, and results for polygons in space follow in a manner entirely analogous to that employed above. One difference as to the character of point symmetries may be noticed: In the plane a point symmetry is identical with a rotation of the plane in itself through 180° ; in space however point symmetry is not equivalent to a rotation since in fact corresponding figures are not congruent but differ in the order of arrangement of their parts. If we consider a central symmetry in the plane as a rotation, the analogue in space would be a line symmetry, which is in fact rotation about an axis through 180° . However, in the application to space polygons only the symmetries of the former type, *i. e.* point symmetries with respect to the vertices, are considered.

The results hold also for one dimension, that is, for sets of points in a line. Thus, for any set $P_1, P_2, \dots, P_n$ there is a derived "inscribed" set consist-

ing of the mid-points of the segments, $P_1P_2, P_2P_3, \dots, P_nP_1$. This set is entirely arbitrary if n is odd, but not if n is even; the characteristics stated in 9 apply almost literally to these special sets of points of the latter type.

COLUMBIA UNIVERSITY, NEW YORK, *January 15, 1903.*

ANALOG OF SYLVESTER'S DIALYTIC METHOD OF ELIMINATION.

By DR. SAUL EPSTEEN.

If the two equations

$$\begin{aligned} a_0x^3 + a_1x^2 + a_2x + a_3 &= 0, \\ b_0x^2 + b_1x + b_2 &= 0, \end{aligned}$$

have a root x_1 in common, we can write

$$\begin{aligned} a_0x_1^4 + a_1x_1^3 + a_2x_1^2 + a_3x_1 &\equiv 0 \\ a_0x_1^3 + a_1x_1^2 + a_2x_1 + a_3 &\equiv 0 \\ b_0x_1^4 + b_1x_1^3 + b_2x_1^2 &\equiv 0 \\ b_0x_1^3 + b_1x_1^2 + b_2x_1 &\equiv 0 \\ b_0x_1^2 + b_1x_1 + b_2 &\equiv 0. \end{aligned}$$

Eliminating $x_1^4, x_1^3, x_1^2, x_1, 1$, we obtain

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \end{vmatrix} = 0.$$

This is the well known dialytic method of elimination of Sylvester.

We can deal with linear differential equations in exactly the same manner. Suppose there are given the equations

$$(1) \quad \alpha_0(x) \frac{d^3y}{dx^3} + \alpha_1(x) \frac{d^2y}{dx^2} + \alpha_2(x) \frac{dy}{dx} + \alpha_3(x) \cdot y = 0,$$

$$(2) \quad \beta_0(x) \frac{d^2y}{dx^2} + \beta_1(x) \frac{dy}{dx} + \beta_2(x) \cdot y = 0.$$